Partitions on the real line

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section Set Theory
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Definition (Cardinal coefficients)

For any $I \subset \mathscr{P}(X)$ let

$$non(I) = min\{|A| : A \subset X \wedge A \notin I\}$$

$$add(I) = min\{|\mathscr{A}| : \mathscr{A} \subset I \wedge \bigcup \mathscr{A} \notin I\}$$

$$cov(I) = min\{|\mathscr{A}| : \mathscr{A} \subset I \wedge \bigcup \mathscr{A} = X\}$$

$\mathbb{L}$ - σ ideal of null sets

Definition

We say that C is completely $\mathbb{I}$-nonmeasurable in D iff for any $\mathbb{I}$-positive relative Borel subset $B \subseteq D$ both sets $B \cap C$ and $B \setminus C$ are $\mathbb{I}$-positive.

Definition (outer, inner envelope)

Let $\mathbb{I}$ σ-ideal and any $D \subseteq \mathbb{R}$ we can define $[D]_{\mathbb{I}} = B$ is **outer envelope** of D iff

1. $D \subseteq B$ and B is a Borel set with $B \setminus D \in \mathbb{I}$ and
2. if $D \subseteq C$ and C is Borel then $B \setminus C \in \mathbb{I}$.

Define $]D[_{\mathbb{I}}$ as an **inner envelope** od D iff $]D[_{\mathbb{I}}= ([D^c]_{\mathbb{I}})^c$.

Fact

If $\mathbb{I}$ is c.c.c. σ-ideal then the outer and inner envelopes exist for any subset of the real line.

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Theorem (Brzuchowski,Cichoń,Grzegorek and Ryll-Nardzewski)

Let $\mathbb{I}$ σ-ideal with the Borel base containing all singletons.
If $\mathcal{A} \subseteq \mathbb{L}$ be any finite point family with $\bigcup \mathcal{A} = \mathbb{R}$.
Then there exists $\mathcal{A}' \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}'$ is $\mathbb{I}$-nonmeasurable.

See: Brzuchowski J., Cichoń J., Grzegorek E., Ryll-Nardzewski C., On the existence of nonmeasurable unions, Bull. Polish Acad. Sci. Math. 1979, 27, 447-448

Theorem (Fremlin, Todorcevic)

Let $\mathcal{A} \subseteq \mathbb{L}$ be any partition of $[0,1]$ onto null sets.
Then for every $\epsilon > 0$ there exists $\mathcal{A}' \subseteq \mathcal{A}$ such that

$$\lambda_*(\bigcup \mathcal{A}') < \epsilon \wedge \lambda^*(\bigcup \mathcal{A}') > 1 - \epsilon.$$

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Theorem (Rałowski and Żeberski)

Assume that no cardinal $\kappa < 2^\omega$ is quasi-measurable.
Assume that $\mathbb{I}$ satisfies c.c.c. Let $\mathcal{A} \subseteq \mathbb{I}$ be a point-finite family such that $\bigcup \mathcal{A} \notin \mathbb{I}$.
Then there exist pairwise disjoint subfamilies $\mathcal{A}_\xi$, $\xi \in \omega_1$ of $\mathcal{A}$ such that each of the union $\bigcup \mathcal{A}_\xi$ is completely $\mathbb{I}$-nonmeasurable in $\bigcup \mathcal{A}$.

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Theorem (Cichoń, Morayne, Ryll-Nardzewski, Żeberski, RR)

Any partition of $\mathbb{R}$ onto meager sets has a subfamily $\mathcal{A}' \subset \mathcal{A}$ s.t. $\bigcup \mathcal{A}'$ doesn't has Baire property.

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Main inspiration

Theorem (Bukovsky)

Every partition of the real line $\mathbb{R}$ onto null sets has subpartition for which union is nonmeasurable.

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Definition

$\mathcal{A}$ has **(closed splitting property)** iff for any $\mathcal{A}' \subseteq \mathcal{A}$ such that $]\bigcup \mathcal{A}'[_{\mathbb{L}} \notin \mathbb{L}$ there exist non-null closed sets $C_0, C_1 \notin \mathbb{L}$ and $\mathcal{A}_0, \mathcal{A}_1 \subseteq \mathcal{A}'$ such that $\mathcal{A}_0 \cap \mathcal{A}_1 = \emptyset$

$$(\forall i \in \{0,1\})(]\bigcup \mathcal{A}_i[_{\mathbb{L}} \setminus C_i \in \mathbb{L} \quad \wedge \quad C_i \subseteq]\bigcup \mathcal{A}_i[_{\mathbb{L}})$$

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$\mathcal{A}$ is **tiny family** iff

$$(\forall B \in \mathbb{L}) \ \bigcup \{A \in \mathcal{A} : \ B \cap A \neq \emptyset\} \in \mathbb{L}.$$

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Theorem

Let $\mathcal{A} \subseteq P(2^\omega) \cap \mathbb{L}$ be any family of pairwise disjoint null subsets of the Cantor space such that $\bigcup \mathcal{A} \notin \mathbb{L}$. Asume that:

1. *$\mathcal{A}$ is regular family,*
2. *$\mathcal{A}$ has closed splitting property,*
3. *$\mathcal{A}$ is tiny family.*

Then there exists subfamily $\mathcal{A}' \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}' \notin \mathbb{L}$ and $]\bigcup \mathcal{A}'[_{\mathbb{L}} = \emptyset$.

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Proof:

Let κ be smallest cardinality of $\mathcal{A}$ s.t. Theorem is false.
$\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ and define σ-ideal

$$(\forall X \in P(\kappa))(X \in \mathscr{L} \leftrightarrow \bigcup_{\alpha \in X} A_\alpha \in \mathbb{L}).$$

Observe that

- if $X \notin \mathscr{L}$ then $]\bigcup_{\alpha \in X} A_\alpha[_{\mathbb{L}} \notin \mathbb{L}$ then $\mathscr{L}$ is c.c.c.c
- $add(\mathscr{L}) = \kappa$

Let $P(\mathcal{A}) = P(\kappa)/\mathscr{L}$ and $(\mathbb{P}_0, \leq)$ where

$$\mathbb{P}_0 = P(\kappa) \setminus \mathscr{L}$$

and

$$(\forall p, q \in \mathbb{P}_0)\,(p \leq q \leftrightarrow p \subseteq q)$$

Define $[\cdot] : \omega^{<\omega} \to \mathbb{P}_0$ s.t.

- $[\emptyset] = \kappa$ and
- $(\forall t, s \in \omega^{<\omega})(t \subset s \to [s] \leq [t])$ and
- $(\forall t, s \in \omega^{<\omega})((t \neq s \wedge |t| = |s|) \to [s] \cap [t] = \emptyset)$ and
- $(\forall t \in T)(\{[t^\frown n] : n \in \omega\}$ is maximal antichain in $\mathbb{P}_0)$ and
- $(\forall t \in \omega^{<\omega})(\lambda(]\bigcup_{\alpha \in [t]} A_\alpha[_{\mathbb{L}}) < 2^{-|t|})$ and
- $(\forall t \in \omega^{<\omega})(\exists C \notin \mathbb{L})$ C is closed and $]\bigcup_{\alpha \in [t]} A_\alpha[_{\mathbb{L}} \setminus C \in \mathbb{L}$.

Because **closed splitting** and $\mathscr{L}$ is c.c.c. σ - ideal on the κ.
Now let $\mathbb{P}$ be a suborder of $\mathbb{P}_0$ countably generated (as a σ-field) by the family $\{[t] : t \in \omega^{<\omega}\}$.

Let $\dot{r} \in V^{\mathbb{P}}$ be the name for the generic real with

$$[t] \Vdash t \subset \dot{r} \text{ for any } t \in \omega^{<\omega}$$

Let $\dot{R} \in V^{\mathbb{P}}$ - name for set of generic reals

$$\Vdash \dot{R} = \bigcap\{] \bigcup_{\alpha\in[t]} A_\alpha[\mathbb{L}\colon t \subseteq \dot{r}\}.$$

By the last condition for $[\cdot]$ we have $\Vdash \dot{R} \neq \emptyset$.

Claim

$(\forall X \in \mathbb{P})\ X \Vdash$ "$\dot{R} \subseteq] \bigcup_{\alpha \in X} A_\alpha [_{\mathbb{L}}$".

Proof. *By induction over Borel complexity in σ-field $\mathbb{P}$.* ■

$(P, \leq)$ which is equivalent in the forcing sense $(\mathbb{P}, \subseteq)$:

$B \in P \leftrightarrow \exists X \in \mathbb{P} B = \bigcup_{\alpha \in X} A_\alpha$

Definition

Let M be countable elementary submodel of large enough structure H_λ ($|\mathscr{P}(P)|^+ \leq \lambda$) containing forcing notion $P \in M$ defined above. Then $x \in 2^\omega$ is M - genereic real iff

$$\{B \in P \cap M : \ x \in B\} \text{ generate the } P \cap M \text{ generic ultrafilter.}$$

Claim

Let M be countable elementary submodel of large enough structure H_λ containing forcing notion $P \in M$ defined above. Then for every $B \in P \cap M$ there exists nonnull Borel subset of the following set:

$$\{x \in B : \ \ X \textit{ is } M \textit{ - generic real } \}.$$

Proof

Let $B \in P \cap M$, P is c.c.c. then is proper.
Find $Q \leq B$ which is $P \cap M$ - generic one.
If $D \in M$ is any dense then $Q \Vdash \dot{G} \cap D \cap M \neq \emptyset$.
Now consider the following set

$$C = B \cap \bigcap \{ \bigcup \{ p : p \in D \cap M \} : \ D \in M \text{ is open dense set } \},$$

then one can see that there exists $X_0 \in \mathbb{P}$ such that $C = \bigcup_{\alpha \in X_0} A_\alpha$ (because P is an σ-algebra as $\mathbb{P}$ is and M is countable).
Let observe that $C = \{x \in B : \ x \text{ is } M\text{-generic real.}\}$
Now we show that $C \notin \mathbb{L}$,
if not then the set $\{q : q \Vdash \dot{R} \subseteq B \setminus C\}$ is dense under B and then $B \Vdash \dot{R} \subseteq B \setminus C$.

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Let $B \in P \cap M$, P is c.c.c. then is proper.
Find $Q \leq B$ which is $P \cap M$ - generic one.
If $D \in M$ is any dense then $Q \Vdash \dot{G} \cap D \cap M \neq \emptyset$.
Now consider the following set

$$C = B \cap \bigcap \{ \bigcup \{ p : p \in D \cap M \} : \ D \in M \text{ is open dense set } \},$$

then one can see that there exists $X_0 \in \mathbb{P}$ such that $C = \bigcup_{\alpha \in X_0} A_\alpha$ (because P is an σ-algebra as $\mathbb{P}$ is and M is countable).
Let observe that $C = \{x \in B : \ \ x \text{ is } M\text{-generic real.}\}$
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From the other side take any $G \ni Q$ - P generic over V. Take any $p \in G \cap M$, find any $q \in G$ such that $q \leq p, Q$. Then $q \Vdash \dot{R} \subseteq q \subseteq p$.
Then $V[G] \models (\forall p \in G \cap M)\dot{R}_G \subseteq p$.
But $q \Vdash \dot{G} \cap M \cap D \neq \emptyset$ for every dense open set $D \in M$.
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Claim (Borel reading names)

Let $\dot{x}$ be any $\mathbb{P}$ - name s.t. $\Vdash \dot{x} \in 2^\omega$. Let us choose any condition $B \in \mathbb{P}$ and let $\dot{s}$ be any $\mathbb{P}$-name s.t. $\Vdash \dot{s} \in \dot{R}$. Then there exists a stronger condition $C \in \mathbb{P}$, Borel nonnull set $D \subseteq]\bigcup_{\alpha \in C} A_\alpha[_{\mathbb{L}}$ and Borel function $f : D \to 2^\omega$ coded in the ground model V such that $f(\dot{s}_G) = \dot{x}_G$ in any generic extension $V[G]$.

Proof. *Let* $B \in P$, $M \prec H_\lambda$ *for large* λ *s.t.* $(2^{|P|})^+ < \lambda$ *with* $P, B \in M$.
Find $C \le B$ *(master condition) by previous Claim.*
Consider an open base $\mathscr{O}$. *Let*

$$f_O^+ = \bigcup\{]p[_{\mathbb{L}} \times O : \quad p \in P \cap M \wedge p \Vdash \dot{x} \in \check{O}\}$$

$$f_O^- = \bigcup\{]p[_{\mathbb{L}} \times 2^\omega \setminus O : \quad p \in P \cap M \wedge p \Vdash \dot{x} \notin \check{O}\}$$

for any $O \in \mathscr{O}$.
M *is countable then* $f = \bigcap_{O \in \mathscr{O}} (f_O^+ \cup f_O^-)$ *is Borel function and* $C \subseteq dom(f)$ *and* $f(r) = x_G$ *where* G *is as in the definition for* M *generic real.* ∎

Remark

For fixed large enough $M \prec H_\lambda$ and any G -P generic $/V$, f is a constant on $\dot{R}_G$

$$\dot{R}_G \subset \bigcap (G \cap M) \text{ and } \dot{R}_G \subset \{x : x \text{ is } M \text{ generic real}\}$$

Claim

$P(\mathcal{A}) \Vdash 2^\omega \cap V$ *is nonnull* .

Proof

If not then $]\bigcup\mathcal{A}[_{\mathbb{L}}\cap V \in \mathbb{L}$ in $V[G]$ but $(\bigcup\mathcal{A} \notin \mathbb{L})^V$. Take a Borel set $B \in \mathbb{L}(2^\omega \times 2^\omega)$ coded in V s.t. $]\bigcup\mathcal{A} \cap V[_{\mathbb{L}} = B_s$ where $s = \dot{s}_G$ with $\Vdash \dot{s} \in \dot{R}$.
Take $x \in]\bigcup\mathcal{A}[_{\mathbb{L}}\cap V$. Then we have

$$x \in B_s \leftrightarrow (s, x) \in B \leftrightarrow s \in B^x$$

and we have $R \subseteq B^x$ (f is constant on $\dot{R}_G$ f-Borel reading names).
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Now let us consider any G - $\mathbb{P}$ generic ultrafilter over V.
$\mathscr{L}$ is κ - complete ideal on κ.
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Consider $j : V \to Ult(V, G)$ elementary embedding, $cp(j) = \kappa$.
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Theorem

Let $\mathcal{A} \subseteq P(2^\omega) \cap \mathbb{L}$ be any family of pairwise disjoint null subsets of the Cantor space such that $\bigcup \mathcal{A} \notin \mathbb{L}$. Asume that:

1. *$\mathcal{A}$ is regular family,*
2. *$\mathcal{A}$ has closed splitting property,*
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Then there exists subfamily $\mathcal{A}' \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}'$ is completely nonmeasurable.

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Claim

Assume that $\mathcal{A}' \subseteq \mathcal{A}$ *s.t.* $[\bigcup \mathcal{A}']_{\mathbb{L}} \neq \mathbb{R}$. *Then*

$$(\exists \mathscr{B} \subseteq \mathcal{A} \setminus \mathcal{A}') \ \bigcup \mathscr{B} \setminus [\bigcup \mathcal{A}']_{\mathbb{L}} \notin \mathbb{L} \ \wedge \ ]\bigcup \mathscr{B}[_{\mathbb{L}}^{\bigcup(\mathcal{A}\setminus\mathcal{A}')} = \emptyset.$$

Thank You

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